PRIME 2022 POSET THEORY RESEARCH GROUP FINAL REPORT

CORY COLBERT, RICARDO GARCIA, MEHEK MEHRA, JOELLE OCHELTREE, RODRIGO SMITH

1. INTRODUCTION

1.1. **Historical Development.** If P is any poset, then the set $\operatorname{Aut} P$ of all automorphisms of P forms a group under function composition. It is natural to wonder about the other direction. That is, given a group G, when is G the automorphism group of some poset? If G is a group and it is isomorphic to $\operatorname{Aut} P$ for some poset P, we will say G is *realizable via P*. In 1946, G. Birkhoff proved in [3] that if G is a finite group of order n, then G is realizable via a poset P with $n^2 + n$ points. In 1972, M. Thornton showed in [5] that if G is a group of order n with a generating set of r elements, then G can be realized via a poset P with $n(2r + 1)$ points. In 2009, J. Barmak and E. Minian improved Thornton's result in [2], where they show that such groups G can be realized with $n(r+2)$ points. In 2020, Barmak dramatically improved Birkhoff's result, proving that G can be realized via a poset with just $4n$ points (see [1]). In the same paper, Barmak made a fascinating remark that in an apparently not well-known German paper authored by R. Frucht in 1939, the author obtained the 2009 Barmak-Minian bound of $n(r+2)$. The results of Birkhoff, Thornton, and Barmak-Minian, were all independently obtained and proved, however.

1.2. **Connections from Order Theory to Algebraic Geometry and Group Theory.** Studying automorphisms of posets is interesting enough in its own right, but there are deep connections to other realms of mathematics, including algebraic geometry and commutative algebra. These connections add even more importance to understanding posets in abstraction from the point-of-view of automorphisms. If R is a commutative ring with 1, then the set $\operatorname{Spec} R = \{P : P \text{ is a prime ideal of } R\}$ can be equipped with a partial order: $P \leq Q \iff P \subseteq Q$. While it is known which posets arise as the spectrum of a commutative ring [4], it is not known which posets arise as the spectrum of a commutative *Noetherian* ring [6]. If σ is a ring automorphism of a Noetherian ring R (or any commutative ring with 1), then σ induces a poset automorphism of $(\operatorname{Spec} R, \subseteq)$. In particular, there is a relationship between the automorphism group of $\operatorname{Aut} R$, the set of automorphisms of the ring R, and $\operatorname{Aut} \operatorname{Spec} R$. In particular, if one wishes to understand which posets can arise as the spectrum of a commutative Noetherian ring, it is useful to consider what limitations might exist on the automorphism groups of their prime ideal spectra.

Another interesting example occurs when one studies the subgroup lattice of a group G. Like the prime ideal spectra referred to above, the set S of all subgroups of a fixed group G is a poset under subset inclusion. An important question that arises when studying G is which, if any, of its proper, nontrivial subgroups are normal. If $g \in G$, then the conjugation map $\sigma_g : G \to G$ given by $\sigma_g(x) = gxg^{-1}$ is an automorphism of G. So it induces a poset automorphism of S. In particular, when viewing S as a poset, the normal subgroups of G correspond exactly to those points in S that have trivial orbit under the action of the inner automorphisms of G on S. Put another way, if P is

a poset that is known to be the subgroup lattice of a group G, and there is a point $N \in P$ that is fixed by every automorphism of the poset P, then N must correspond to a normal subgroup of G.

In particular, by understanding the automorphism groups of posets, which includes a study of which groups are realizable, one can potentially make interesting statements about related algebraic and algebro-geometric structures.

1.3. **Summary of Results.** In this report, we tackle certain infinite groups and study their realizability. Specifically, after establishing some preliminaries and notation, we study finitely-generated abelian groups and free groups, and we obtain the following results concerning their realizability:

Theorem 1.1. Every finitely generated abelian group is realizable.

Theorem 1.2. Every finitely generated free group is a subgroup of the automorphism group of some poset.

We also study $\beta(G)$, defined in [1], to be the least number of points needed in a nonempty poset P to realize G as $\operatorname{Aut} P$. Specifically, we show:

Proposition 1.3. If G and H are finite groups, then $\beta(G \times H) \leq \beta(G) + \beta(H)$ and $\beta(S_n) = n$ for all $n \in \mathbb{N}$.[1]

Finally, we study how to count automorphisms with Python by associating, to any finite poset P, a useful matrix C that we use to count automorphisms of P.

2. Definitions and Preliminaries

Definition 2.1. A poset $(X, \leq_X)$ is a set X with a relation $\leq_X$ such that for all $x, y, z \in X$, we have $x \leq_X x$; $x \leq_X y$ and $y \leq_X x \implies y = x$; and $x \leq_X y$ and $y \leq_X z \implies x \leq_X z$.

Definition 2.2. If P is a poset and $x, y \in P$, we say y *covers* x in P if $x <_P y$ and for all $z \in P$ such that $x \leq z \leq y$, we have $z = x$ or $z = y$.

Definition 2.3. If P is a poset and $Q \subseteq P$, we say Q is a subposet of P if it is a poset and $x \leq_Q y$ iff $x \leq_P y$.

Definition 2.4. A chain C in a poset P is a subset of P such that for all $x, y \in C$ we have $x \leq_P y$ or $y \leq_P x$. If C is a finite chain, we define its length $\ell(C)$ to be $|C| - 1$.

Definition 2.5. If X is a poset, we define its dimension to be $\sup\{\ell(C) : C$ is a chain in $X\}$. If $x \in X$ is a point, we define $\operatorname{ht}_X(x) := \dim L_X(x)$ and we define $\operatorname{coht}_X x := \dim G_X(x)$.

Definition 2.6. A poset P is *connected* if for all $x, y \in P$ there exists a sequence $(x = \alpha_1, \alpha_2, ..., \alpha_n = y)$ of points $\alpha_i \in P$ such that α_i is comparable to α_{i+1} for all $1 \leq i < n$.

Definition 2.7. Let $\{P_i\}_{i \in I}$ be a collection of posets with pairwise-disjoint sets of points. The disjoint union $\mathcal{P} := \sqcup P_i$ is a poset with order $x \leq_{\mathcal{P}} y$ if and only if $x \leq_{P_i} y$ for some $i \in I$.

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[1]The first part of Proposition 1.3 was, unbeknownst to us, mentioned in a question on Math StackExchange by user "h4tter" in 2019 (see https://math.stackexchange.com/q/3123314). Our independently-obtained construction for 1.3 is very slightly different than what is in the StackExchange question, but it does have a lot of similarities.

Definition 2.8. We say $f : P \to Q$ is a poset map if for all $x, y \in P$, we have $x \leq_P y \implies f(x) \leq_Q f(y)$. A poset map $f : P \to Q$ is said to be order-reflexive if for all $x, y \in P$, whenever $f(x) \leq_Q f(y)$, we have $x \leq_P y$. A surjective, order-reflexive map $f : P \to Q$ is called a poset isomorphism. An isomorphism $f : P \to P$ is called an automorphism. The set $\operatorname{Aut} P$ of all automorphisms from P onto P is a group under composition.

Definition 2.9. A group G is realizable if there exists a poset P such that $\operatorname{Aut} P = G$.

Definition 2.10. If G is a finite group, define $\beta(G) = \min\{|P| : P \text{ is a poset with } \operatorname{Aut} P \cong G\}$.

Definition 2.11. If P is a poset and $p \in P$, we define the *orbit* of p to be the set $\{f(p) : f \in \operatorname{Aut} P\}$.

Definition 2.12. If P is a poset, we say $K \subseteq P$ is a connected component of P if K is a subposet of P and it is maximal, with respect to set-theoretic inclusion of subposets, with respect to being connected.

3. Results

3.1. **On Finitely Generated Abelian Groups and Free Groups.**

Lemma 3.1. If $f : P \to Q$ is an isomorphism and K is a connected component of P, then $f(K)$ is a connected component of Q. In particular, if $f \in \operatorname{Aut} P$, then f carries connected components to connected components.

Proof. Let $x, y \in f(K)$ and write $x = f(x')$ and $y = f(y')$ for $x', y' \in K$. Since K is connected, there is a path $(x', x_1, \ldots, x_n, y')$ in K from x' to y'. So $(f(x'), \ldots, f(y'))$ is a path in $f(K)$ from x to y. So $f(K)$ is connected if K is. If $f(K) \subseteq J$ for some connected J in Q, then $f^{-1}(J)$ must also be connected in P and contain K. So $f^{-1}(J) = K$ by maximality of K. Thus, $f(K) = J$ since f is a bijection. $\square$

Lemma 3.2. Let $P_1, \ldots, P_k$ be a sequence connected, pairwise disjoint and pairwise non-isomorphic posets. Let $\mathcal{P} := \sqcup_{i=1}^k P_i$ be the disjoint union of the P_i. Then $\operatorname{Aut} \mathcal{P} \cong \prod_{i=1}^k P_i$.

Proof. Let $f \in \operatorname{Aut} \mathcal{P}$. If $1 \leq i \leq k$, then $f(P_i)$ is a connected component of P by Lemma 3.1. So $f(P_i) = P_j$ for some j since the set of connected components of $\mathcal{P}$ is precisely $\{P_1, \ldots, P_k\}$. Since f is an automorphism of P, we have $P_i \cong f(P_i) = P_j$, so $i = j$ by assumption. In particular, $f|_{P_i} \in \operatorname{Aut} P_i$. For simplicity, if $f \in \operatorname{Aut} \mathcal{P}$, define $f_i \in \operatorname{Aut} P_i$ as $f_i := f|_{P_i}$.

The map $\varphi : \operatorname{Aut} \mathcal{P} \to \prod_{i=1}^k \operatorname{Aut} P_i$ via $\varphi(f) = (f_1, \ldots, f_k)$ is clearly an injective mapping of groups, and if $(g_1, \ldots, g_k) \in \prod_{i=1}^k \operatorname{Aut} P_i$, then $g := \cup_{i=1}^k g_i$ is an automorphism of P with $\varphi(g) = (g_1, \ldots, g_k)$, so φ is surjective hence an isomorphism. $\square$

If P is any poset, it is possible to add a "single minimal node" to the bottom of P by simply setting $Q = P \cup \{\alpha\}$ where $\alpha \notin P$, and declaring $\alpha <_Q p$ for all $p \in P$ and of course $\alpha \leq_Q \alpha$.

Lemma 3.3. If P is any poset and $Q = P \cup \{\alpha\}$ where α is such that $\alpha <_Q p$ for all $p \in P$, then $\operatorname{Aut} Q \cong \operatorname{Aut} P$.

Proof. Every automorphism of Q must fix α so the restriction map $f \to f|_P$ is an isomorphism from $\operatorname{Aut} Q$ onto $\operatorname{Aut} P$. $\square$

Theorem 3.4. Every finitely generated abelian group is the automorphism group of a poset.

Proof. Let G be a finitely generated abelian group, and write $G = \mathbb{Z}^r \times T$, where T is a finite group and $r > 0$ is a positive integer. Define $P_0 := \mathbb{Z}$ with the usual order. Let $n > 0$, and suppose P_m has been defined for $0 \leq m < n$. Define $P_n = P_{n-1} \cup \{\alpha\}$, where $\alpha < p$ for all $p \in P_{n-1}$. Let Q be a poset with automorphism group T, which exists by [2]. Consider $\mathcal{P} = \sqcup_{i=1}^r P_i \sqcup Q$. Now $P_i \not\cong P_j$ for $i \neq j$ because each P_m has precisely m points of finite height by construction. Moreover, $Q \not\cong P_i$ for any $1 \leq i \leq r$ because $\operatorname{Aut} Q$ is finite while $\operatorname{Aut} P_i \cong \mathbb{Z}$ is infinite. In particular, by Lemma 3.2, $\operatorname{Aut} \mathcal{P} = \mathbb{Z} \times \cdots \times \mathbb{Z} \times T = G$. $\qquad\square$

Let $S = \{g_1, \ldots, g_n\}$ be a nonempty, finite set, and let F_S be the free group on S. Let $U = S \cup \{1_{F_S}\}$. For $x, y \in F_S$, declare $x \leq_{F_S} y$ if and only if $y = xh_1 \cdots h_k$ for some $h_1, \ldots, h_k \in U$.

Proposition 3.5. With the above construction, $(F_S, \leq_{F_S})$ is a poset.

Proof. For simplicity, write $F = F_S$. If $x \in F$, then $x = x \cdot 1_F$, so $x \leq_F x$ and $\leq_F$ is reflexive. If $x \leq_F y$ and $y \leq_F x$, then $y = xh_1 \cdots h_k$ and $x = yt_1 \cdots t_s$ for some $h_1, \ldots, t_s \in U$. Therefore, $1 = h_1 \cdots h_k t_1 \cdots t_s$, so $h_i, t_j = 1$ for all i, j since each h_i, t_j is an element of S or is the identity. In particular, $x = y$. Lastly, if $x \leq_F y$ and $y \leq_F z$, then $y = xh_1 \cdots h_k$ and $z = yr_1 \cdots r_l$, so $z = xh_1 \cdots h_k r_1 \cdots r_l$. That is, $x \leq_F z$. $\qquad\square$

Theorem 3.6. With F_S as above, we have $F_S \leq \operatorname{Aut} F_S$ as groups. In particular, every finitely generated free group is a subgroup of the automorphism group of some poset.

Proof. As above, write $F = F_S$. Let $x \in F$, and define $\sigma : F \to F$ as $\sigma_x(z) = xz$. If $a \leq_F b$, then $b = ah_1 \cdots h_k$ for some $h_i \in U$. So $xb = xah_1 \cdots h_k$ and thus $xa \leq_F xb$. So σ_x is a poset map. Conversely, if $xa \leq_F xb$, then $xb = xah_1 \cdots h_k$ so $b = ah_1 \cdots h_k$ by the cancellation law. Therefore, $a \leq_F b$ and σ_x is order-reflexive. Finally, if $w \in F$, then $w = xx^{1-}w = \sigma_x(x^{-1}w)$, so σ_x is surjective. Therefore, $\sigma_x \in \operatorname{Aut} F$. The map from F to $\operatorname{Aut} F$ given by $x \to \sigma_x$ is an injective homomorphism. So $F \leq \operatorname{Aut} F$ as groups. $\qquad\square$

3.2. **On Beta Values for Various Finite Groups.** Recall that if G is a finite group, then $\beta(G)$ is the smallest number of points needed in a nonempty poset P to realize G via P.

Proposition 3.7. We have $\beta(S_n) = n$ for all $n \in \mathbb{N}$.

Proof. Let P be a poset with $\operatorname{Aut} P = S_n$, and write $k = |P| > 0$. Every automorphism of P is a bijection of P with itself, so $\operatorname{Aut} P = S_n \leq S_P = S_k$. Therefore, $n! \leq k!$ so $n \leq k$. So $\beta(S_n) \geq n$. If Q is an antichain with n points, then $\operatorname{Aut} Q \cong S_n$, so there is a poset with exactly n points whose automorphism group is S_n. Thus, $\beta(S_n) = n$. $\qquad\square$

If P and Q are posets with disjoint sets of points, define $P^Q := P \cup Q$ with the following order: $x \leq_{PQ} y$ if and only if $x \leq_P y$ or $x \leq_Q y$ or $x \in P$ and $y \in Q$.

Lemma 3.8. If P and Q are finite posets, then $\operatorname{Aut} P^Q = \operatorname{Aut} P \times \operatorname{Aut} Q$.

Proof. Let $n = \dim P$ and $m = \dim Q$, and if X is any poset, let H_i^X be the set of points in X whose height is i in X. We claim $P = \cup_{i=0}^n H_i^{P^Q}$, and $Q = \cup_{j=n+1}^{n+m} H_j^{P^Q}$. If $p \in P$, then $\operatorname{ht}_{PQ} p \geq \operatorname{ht}_P p$ by definition. If $x \leq_{PQ} p$, then we must have $x \in P$ by definition, so $\operatorname{ht}_{PQ} p \leq \operatorname{ht}_P p$. In particular, $P \subseteq \cup_{i=0}^n H_i^{P^Q}$. If $p \notin P$, then $p \in Q$. Let $r \in P$ be a point whose height is n in P. Then $r < p$ by definition, so $\operatorname{ht} p \geq n + 1$. That is, $p \notin \cup_{i=0}^n H_i^{P^Q}$. So $P = \cup_{i=0}^n H_i^{P^Q}$. A similar argument shows $Q = \cup_{j=n+1}^{n+m} H_j^{P^Q}$, so in fact $\dim P^Q = n + m$.

Let $f \in \operatorname{Aut} P^Q$. Since f is an automorphism, we have $f(H_i^{P^Q}) = H_i^{P^Q}$ for all i. In particular, f restricts to an automorphism of the set of all the points in P^Q whose height is at most n. That is, f restricts to an automorphism of P. Likewise, f restricts to an automorphism of Q. So we get a map from $\operatorname{Aut} P^Q$ to $\operatorname{Aut} P \times \operatorname{Aut} Q$ via $f \to (f|_P, f|_Q)$. It is clearly an injective homomorphism of groups. To see why it is surjective, suppose $(g, h) \in \operatorname{Aut} P \times \operatorname{Aut} Q$, and define $f = g \cup h$. Suppose $x <_{PQ} y$, and $x \in P$ and $y \in Q$. Then $f(x) = g(x) \in P$ and $f(y) = h(y) \in Q$. So $f(x) = g(x) <_{PQ} h(y) = f(y)$ by definition of the order on P^Q. Conversely, if $f(x) <_{PQ} f(y)$, and $f(x) \in P$ and $f(y) \in Q$, then $f(x) = g(x')$ for $x' \in P$ and $f(y) = h(y')$ for $y' \in Q$. So $x' <_{PQ} y'$ by definition. Surjectivity of f is clear. So f is an automorphism of P^Q. $\qquad\square$

Proposition 3.9. If G and H are finite groups, then $\beta(G \times H) \leq \beta(G) + \beta(H)$.

Proof. Suppose $G \not\cong H$, and let P_G be a poset with $\beta(G)$ points such that $\operatorname{Aut} P_G \cong G$, and let P_H be a poset with $\beta(H)$ points such that $\operatorname{Aut} P_H \cong H$. Set $\mathcal{P} = P_G \sqcup P_H$. Then $P_G \not\cong P_H$ because $G \not\cong H$. So $\operatorname{Aut} \mathcal{P} = \operatorname{Aut} P_G \times \operatorname{Aut} P_H = G \times H$ by Lemma 3.2. Moreover, $|\mathcal{P}| = \beta(G) + \beta(H)$. So $\beta(G \times H) \leq \beta(G) + \beta(H)$.

In the other case, where $G \cong H$, consider $P_G^{P_H}$. By Lemma 3.8, $\operatorname{Aut} P_G^{P_H} \cong \operatorname{Aut} G \times \operatorname{Aut} H$ and of course $|P_G^{P_H}| = \beta(G) + \beta(H)$. $\qquad\square$

Remark. We could have taken $P_G^{P_H}$ from the outset, but we wished to show an alternative approach to arguing with disjoint unions, which, from a technical point of view, is at least an easier order to establish. Moreover, the dimension of $P_G \sqcup P_H$ is the larger of $\dim P_G$ and $\dim P_H$, whereas with $P_G^{P_H}$, the dimension is the sum of the dimensions of P_G and P_H.

3.3. **Counting Automorphisms with Python.** To count the automorphisms of a given poset P, we developed an algorithm in Python that works by first generating the symmetric group S_P and then checking which elements of S_P are automorphisms of P. To check if the elements of the symmetric group are automorphisms, the program uses a matrix representation of P. Specifically, if we enumerate $P = \{0, 1, \ldots, n\}$ for some $n \geq 1$, we define a matrix $C = [c_{ij}]$ where $c_{ij} = 1$ if and only if $i = j$ or node j covers node i in the poset. Otherwise, set $c_{ij} = 0$. Given a bijection $f \in S_P$, the program applies f to C by swapping rows and columns based on f to form matrix $D_f = [d_{ij}]$ given by $d_{ij} = c_{f(i)f(j)}$ for all $0 \leq i, j \leq n$. With this setup, we have:

Theorem 3.10. Let $P = \{0, 1, \ldots, n\}$ be a poset, and let $f \in S_P$. Then $f \in \operatorname{Aut} P$ if and only if $D_f = C$.

Proof. Let $f \in \operatorname{Aut} P$, and let $i, j \in P$ be distinct points. Either j covers i in P or not. If j covers i, then $f(j)$ covers $f(i)$ since f is an automorphism. Therefore, $d_{ij} = c_{f(i)f(j)} = 1$. Since j covers i, $c_{ij} = 1$. So $c_{ij} = d_{ij}$. If j does not cover i, then $f(j)$ cannot cover $f(i)$, so $c_{ij} = 0 = c_{f(i)f(j)} = d_{ij}$. So $D_f = C$.

Conversely, if $D_f = C$, we claim $f \in \operatorname{Aut} P$. Suppose $i \leq_P j$, and write $i := i_0 <_P i_1 <_P i_2 <_P \ldots <_P i_k := j$, where each i_r is covered by i_{r+1}. Then $c_{i_r i_{r+1}} = 1$ for all $1 \leq r < k$, so $1 = c_{i_r i_{r+1}} = d_{i_r i_{r+1}} = c_{f(i_r)f(i_{r+1})}$ for all such r. Therefore, $f(i_{r+1})$ covers $f(i_r)$ for all r and so $f(i) <_P f(j)$ by transitivity of the relation on P. That is, f is a poset map. Conversely, suppose $f(i) <_P f(j)$. As before, $f(i) := t_0 < t_1 <_P t_2 <_P \ldots <_P t_s := f(j)$ for some $t_i \in P$ where each t_{w+1} covers t_w. Since f is surjective, $t_w = f(i_w)$ for some $i_w \in P$, so $f(i_0) <_P f(i_1) <_P \ldots <_P f(i_s)$

and since $D_f = C$, we have $c_{i_w i_{w+1}} = d_{i_w i_{w+1}} = c_{f(i_w)f(i_{w+1})} = 1$ for all w. So i_{w+1} must cover i_w for all w. So $i <_P j$, which implies that f is order-reflexive. Since f is surjective, $f \in \operatorname{Aut} P$.   □

Hence, the program generates all of the bijections that result in the original matrix are automorphisms on the poset. The program then counts all such bijections and returns them to the user. The code is available on the GitHub repository at `rb.gy/cnzpfc`

4. Future Work

There is much work left to do. First, we would understand $\beta(Z_n)$ for all cyclic groups Z_n, or at least a wider class than is known. Perhaps the easiest such class is cyclic groups of the form Z_{p^k} where p is a prime number and $k > 0$ is an integer. In 2020, Barmak provided some upper and lower-bounds of $\beta(G)$ for these groups G. Specifically, Barmak showed in [1]:

Theorem 4.1. (Barmak, '20) Let p be a prime number, and let $k \geq 2$ be an integer.

   (i) $\beta(Z_2) = 2$.
   (ii) $2^{k+1} \leq \beta(Z_{2^k}) \leq 2^{k+1} + 12$.
   (iii) $2p^k \leq \beta(Z_{p^k}) \leq 2p^k + 3p$, if $p = 3, 5$.
   (iv) $2p^k \leq \beta(Z_{p^k}) \leq 2p^k + p$, if $p \geq 7$.

Barmak's work along with computational data leads us to the following conjecture:

Conjecture 4.2. We have $\beta(Z_{p^k}) = 2p^k + p$ for all primes $p \geq 7$ and $k \geq 1$.

In addition, we also wish to continue our efforts to discover more infinite groups that arise as automorphism groups of posets. Specifically, we believe all finitely generated free groups are realizable. We also wish to explore what happens with the Frucht-Barmak-Minian construction if you replace the finite group G with an infinite one.

Lastly, while the Python code in the previous section served our internal purposes excellently, we have a strong interest in making the code as efficient as possible by using facts about automorphisms. We already know that every automorphism must preserve height, and that greatly narrows the search from all bijections to only a relative handful. But there are still many automorphisms to check for a given poset P. Even with this consideration, there are still as many as $\prod_{i=0}^{k} n_i!$ automorphisms of P, where n_i is the number of points in P whose height is i, and $k = \dim P$.

5. Acknowledgements

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